Modeling lightning as a 2-dimensional fractal in an electric field

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Abstract:

Many natural phenomena, including tree branches, snowflakes, coastlines, and lightning, can be modeled using fractal patterns. A fractal is a mathematical set that uses a recursive formula to produce an often self-similar shape with a nonintegral dimension. This experiment uses a common algorithm in lightning research, known as the dielectric breakdown model, to model a lightning strike as a fractal. This approach is similar to diffusion-limited aggregation, another method used to generate fractals, and we have created our own adaptation of a diffusion-limited aggregation to compare to the dielectric breakdown model for lightning. The largest criterion for comparison is the fractal dimension generated by each model. We will then use our models investigate the behavior of our lightning strike when it occurs near various potential patterns on the ground, imitating lightning rods, trees, or buildings.

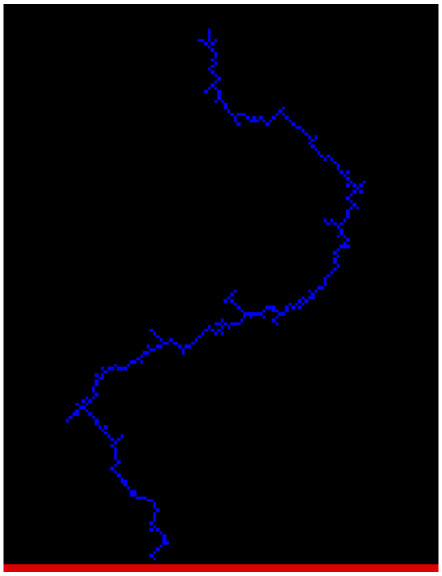


Image 1: Our lightning simulation   Image 2: Lightning photograph

http://www.windsun.com/Photovolaic_Systems/Lightning_Protection.htm

Introduction to Lightning:[1]

During an electrical storm, clouds become highly polarized, where the upper portion of the cloud is positive and the lower portion is negative. Scientists do not completely agree on how this separation occurs, but for further information and one plausible explanation, see reference 1. Once the electric dipole in the cloud grows sufficiently strong, the resulting electric field causes the surrounding air to "break down," beginning a path for current to flow and neutralize the charge separation. Once the electric field reaches a magnitude on the order of tens of thousands of volts per inch, the electric field ionizes the air, separating the positive and negative charges in the air. In lightning terminology, this ionized air is known as plasma, and the plasma is a much better conductor of electrons than normal air, allowing for the flow of electrical current.

In other words, the ionized air generates a path for the charge in the storm cloud to propagate through the air in the form of a lightning bolt. Typically, many of these paths form around a storm cloud, and each of these paths is called a step leader, because each acts as a guide for the growth of the lightning.

The shortest distance between two points is a straight line, so it fits that lightning would strike in a straight bolt. However, anyone who has ever watched a lightning storm knows this is definitely not the case. Dust, impurities, or other obstacles may cause the air to ionize more easily in certain directions. Likewise, the electric field caused by the charge buildup in the storm cloud is probably not uniform, so the air will not ionize equally in all directions. Accordingly, the step leaders will follow the path of least resistance, branch off in multiple directions, and continue growing until one of the step leaders reaches the target point of contact. Once one of the step leaders reaches a site where the charge buildup can be discharged, such as the earth or another cloud, massive amounts of electrical current flow through the lightning bolt.

Image 3: Lightning photograph from http://www.rexwallpapers.com/wallpaper/Lightning-6/

We have discussed the propagation of lightning via step leaders from the source of negative charge in a storm cloud, but the role of the site preparing to be struck by lightning is also worth noting. As step leaders propagate toward the earth, scientists have observed that objects “reach out” to the approaching step leaders by growing positive streamers. You can think about these positive streamers as receptor sites for the lightning—when a step leader meets a streamer, the conducting path is complete and the lightning discharges enormous amounts of current. When you stand underneath storm clouds and feel the hair on your arms rise, you are likely producing positive streamers—but don’t fret too much; everything on the earth’s surface can send positive streamers, and the streamers do not continue to grow longer once produced. Positive streamers also tend to be more prominent on sharp edges, such as the point of a lightning rod.

Dielectric Breakdown Model:

The dielectric breakdown model (DBM) is a common method for modeling lightning (see references 2 and 3), and it establishes a grid with regions of varying electric potential to simulate the electrical charge buildup during a lightning storm. To simulate the source of the lightning in the cloud, we place a small region of negative charge, electric potential $\phi = 0$, near the top of the grid. Along the bottom of the grid, we set the boundary condition $\phi = 1$ to simulate the positive charge on the ground. All other grid points are given an initial electric potential equal to 0.5. Then, we allow the physical system to react to the presence of the charged areas by iterating the Laplace equation ten times (our arbitrary choice) over the grid. Laplace's equation is as follows:

$$\nabla^2\phi = 0$$

Computationally, we solve the Laplace equation for a point on the grid by calculating the average potential of the four neighboring grid points:

$$\phi_{i,j} = ¼ * (\phi_{i-1,j} + \phi_{i+1,j} + \phi_{i,j-1} + \phi_{i,j+1})$$

After solving this equation ten times, we examine all the grid points that are adjacent to a negative charge ($\phi = 0$). We will choose one of these points as the new growth site for the lightning, representing a point where the air is ionized and a step leader path for the lightning will develop. Remember that the step leaders follow a path where the electric field is able to ionize the air, which depends on the strength of the electric field at that point as well as unpredictable factors, such as dust or other inconsistencies in the air. To account for the effect of the electric field strength, we assign each of these grid points a probability according to the electric potential at that site. The probability that the i^{th} point will be chosen as the new growth site for a step leader is calculated by the following equation:

$$p(i) = \frac{(\phi_i)^\eta}{\sum_{j \in N} (\phi_j)^\eta}$$

where p(i) is the probability of the i^{th} point being chosen, N is the total number of points which could possibly be chosen, and η is a user-defined parameter which affects the amount of branching in the lightning. Note that in this equation we use the true value of our potential function, rather than the potential difference, which is actually the important parameter for electrical calculations. However, in this case the potential of our lightning bolt equals zero, so the potential difference at each point is simply equal to the value of our potential function at that point.

To choose the site to add to our lightning, we multiply the probability for each grid point by a random number between 0 and 1. This accounts for the randomness of step leader propagation introduced by the inconsistencies, such as dust or other obstacles, in the air during a lightning strike. Then, the grid point for which this value is largest is added to our growing lightning, and we assign this point a potential $\phi = 0$. Then, we again iterate Laplace's equation and repeat the algorithm to continue adding new step leader growth sites.

In our probability equation, we can alter the influence of electric potential by changing the user-defined parameter η. Increasing η increases the weight of electric potential for each point and thus reduces the randomness of the lightning growth. Thus, as we increase η, we expect to see lightning bolts that are generally straighter with less branching. This result is observed in our model, as seen in the following screenshots.

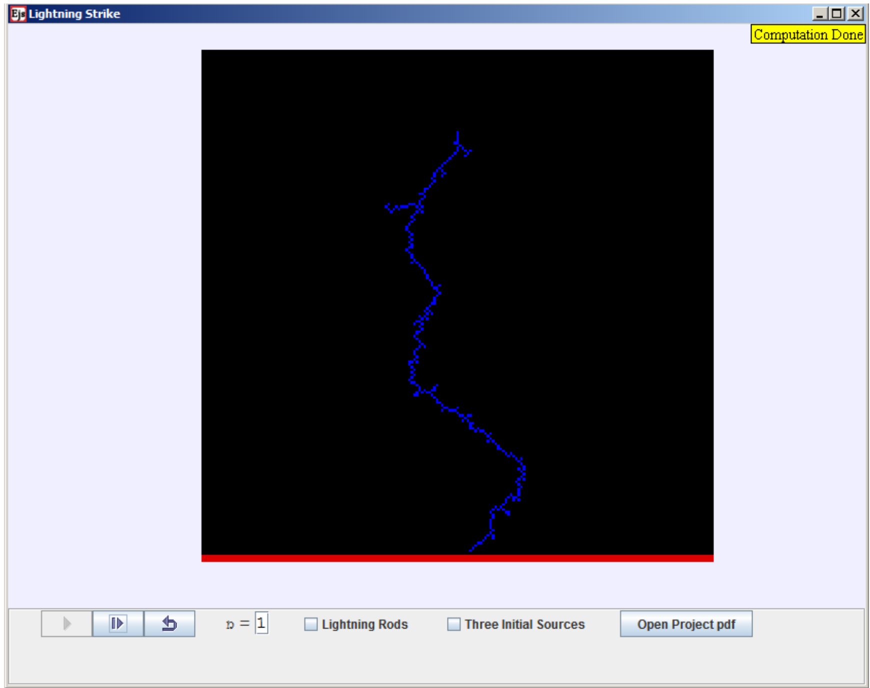

Image 4: DBM simulation with η = 1

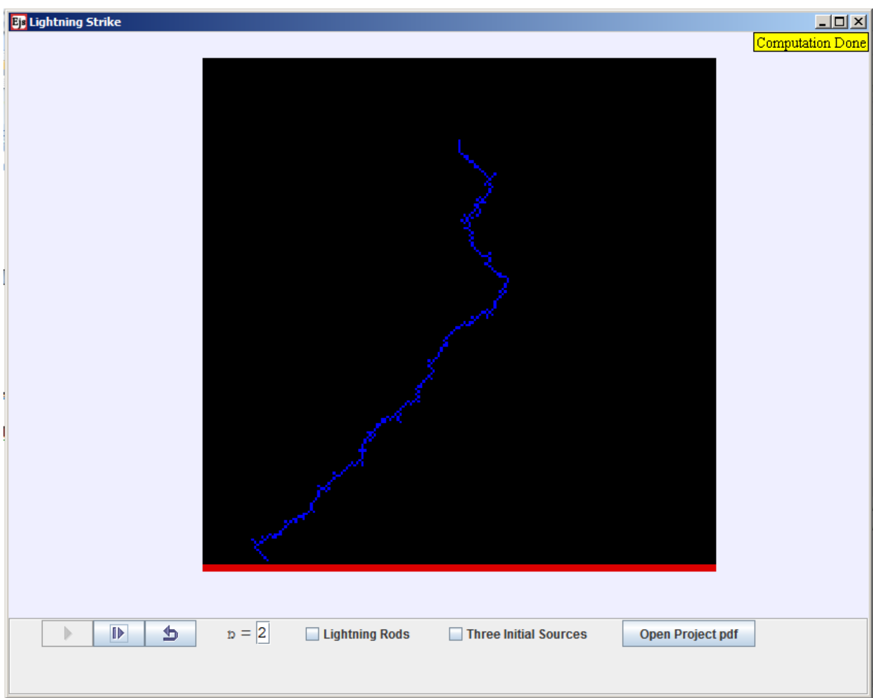

Image 5: DBM simulation with η = 2

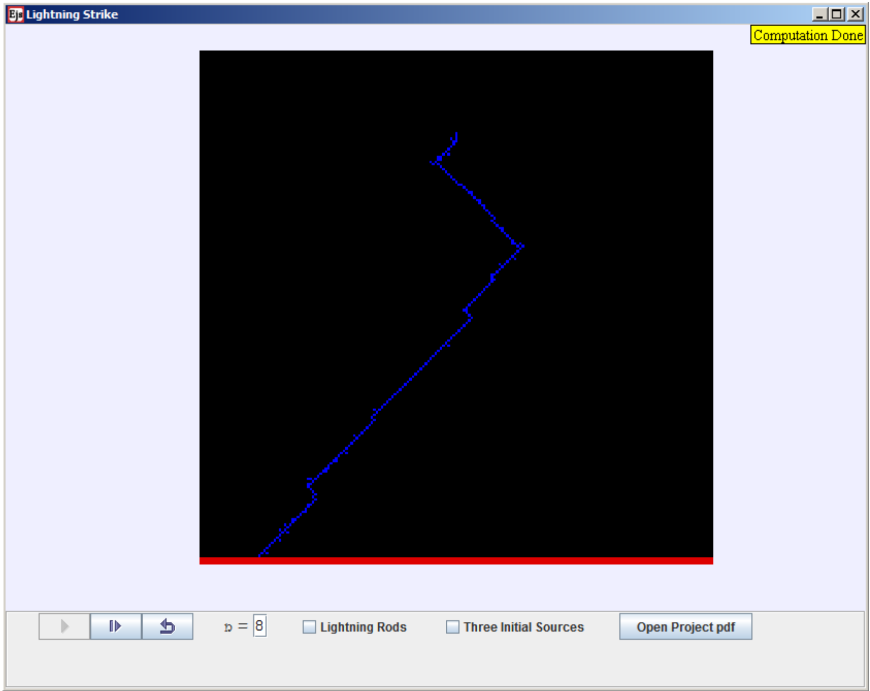

Image 6: DBM simulation with η = 8

Diffusion-limited Aggregation:

Diffusion-limited aggregation (DLA) models fractal growth by allowing a random walker, initially positioned at some point along the perimeter of a circle, to collide with a stationary seed particle. When the walker collides with the stationary aggregate, the walker sticks, and a new walker is positioned on the perimeter of a circle outside the growing aggregate. This walker moves randomly until it, too, sticks to the stationary particles. The process repeats in this way. For more information on DLA, see references 4 and 5.

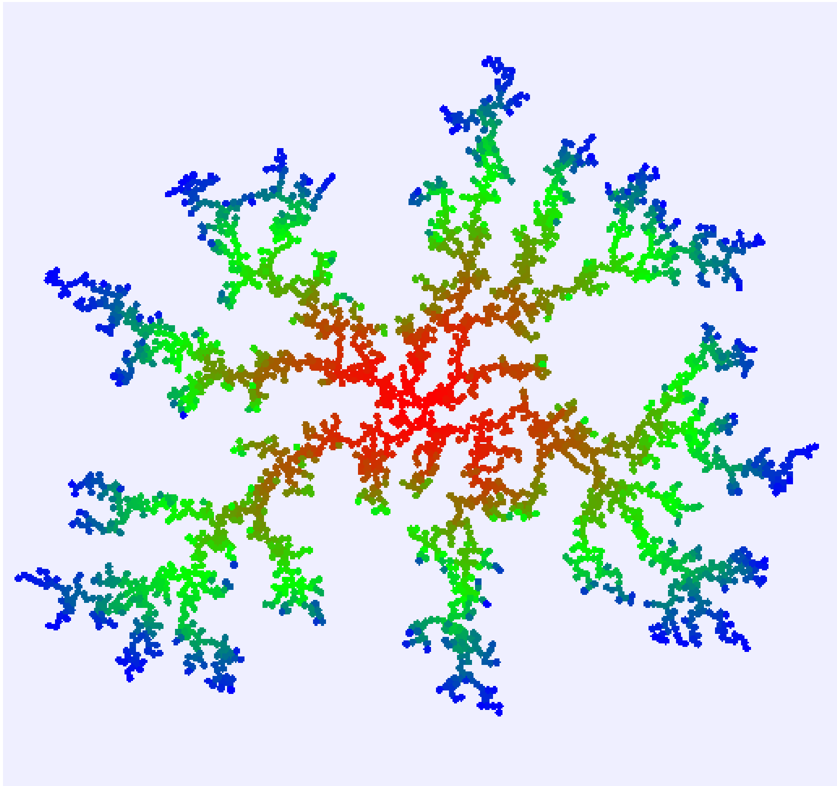

Image 7: A standard DLA cluster, consisting of 2621 particles

To adapt this model to simulate lightning, we restricted the starting positions of the new walkers to lie within 10 degrees in either direction from the line pointing vertically downward from the seed. So, as the lightning grows and each new particle starts further away from the seed, each particle has a greater range of possible values for initial x position. That is, for a given angle θ, as the radius of a circle increases, the initial x position (radius*cos(θ)) and initial y position (radius*sin(θ)) increase. The range of possible starting positions of each new walker is shown in the following diagram.

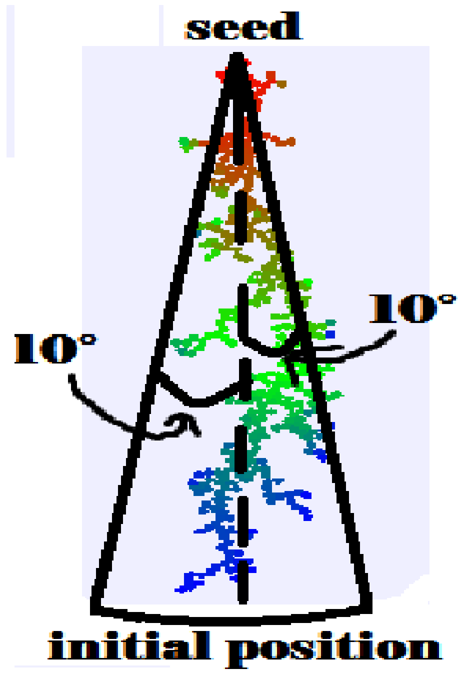

Image 8: Each new walker starts at a random spot along the arc labeled “initial position”

By limiting this angle, we attempt to simulate the influence of positive streamers during a lightning strike. In a storm, the electric field is strongest at points closest to the preparing storm cloud. That is, the electric field is strongest on tall objects or at points on the ground directly beneath the cloud. So, positive streamers will grow most abundantly directly beneath the storm cloud. The step leaders of the lightning, then, are most likely to connect with a positive streamer located near the area directly beneath the storm cloud. By restricting the starting ranges of our new particles, we attempt to model the behavior caused by the prominence of positive streamers directly below the cloud. By allowing each new particle to start at a random position within a range of possibilities, we account for some of the randomness due to inconsistencies in the air and ground.

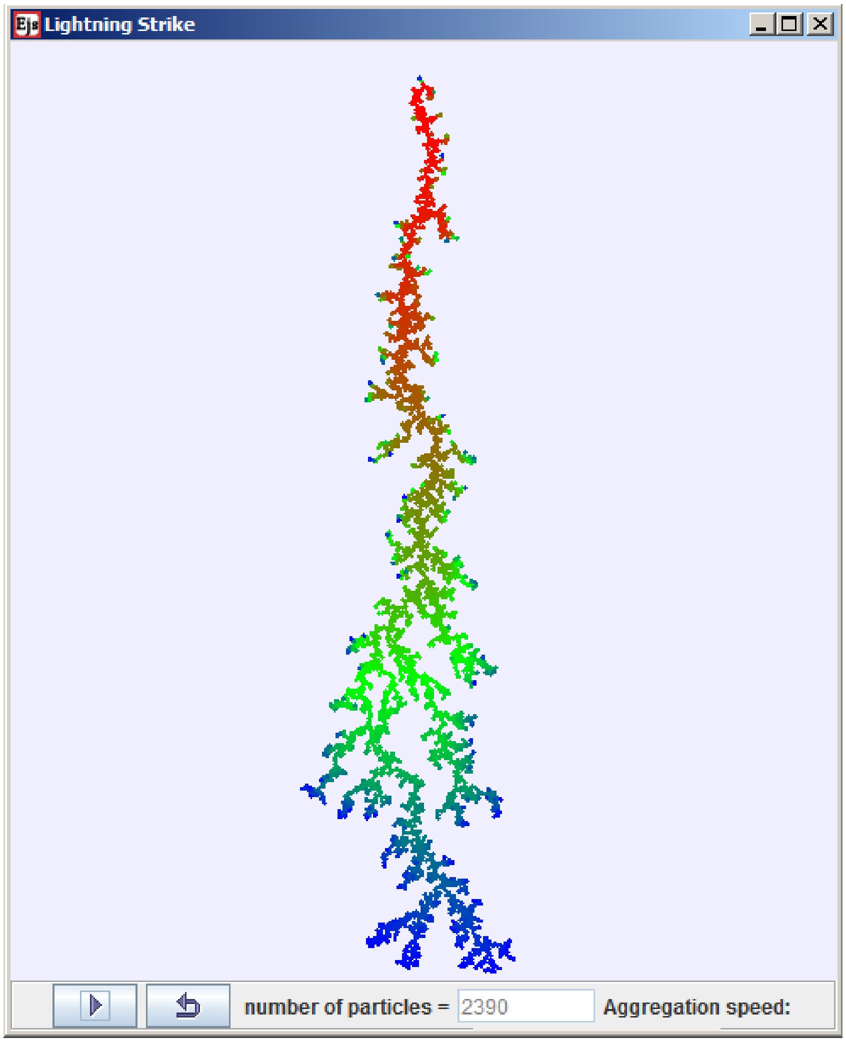

Image 9: DLA model of a lightning strike, consisting of 2390 particles

Fractal Dimension:

The most common way to analyze fractals is to compute their fractal dimension. For a worthwhile discussion on the conceptual meaning of fractal dimension, see reference 6. For our lightning models, we chose to use the radius of gyration method for calculating fractal dimension (reference 7).

The radius of gyration is a root mean square calculation. First, we find the center of mass of the fractal by computing the average distance between each particle and the original seed particle. This is a vector calculation, yielding an x and a y component for the center of mass. For example, for each particle we find the difference between the x position of this particle and the x position of the seed particle. Find the sum of all these distances and divide by the total number of particles. This result is the x component of the center of mass for the fractal (as we assume that each particle has identical, uniform mass).

Then, find the sum of the distance squared between each particle on the fractal and the center of mass. The radius of gyration, R_g, is equal to the square root of this value. For a fractal cluster, R_g is proportional to the number of particles, n, raised to the exponent β. The exponent β is equal to the reciprocal of the fractal dimension. So, we plot $\ln(R_g)$ vs $\ln(n)$ for many different instances, and the slope of the linear fit for this plot is equal to the fractal dimension. For further information, see reference 7.

First, we applied the radius of gyration method to compute the fractal dimension for the DBM simulation. Since the DBM method is well-supported in literature (references 2 and 3), we will compare the dimension of lightning found using this model to the dimension we found using the DLA simulation. If the DLA model produces a dimension similar to that of the DBM, then it is likely that our modified DLA is a good model for lightning. For the DBM, we typically found the dimension to be near the range 1.4-1.5, for all values of η.

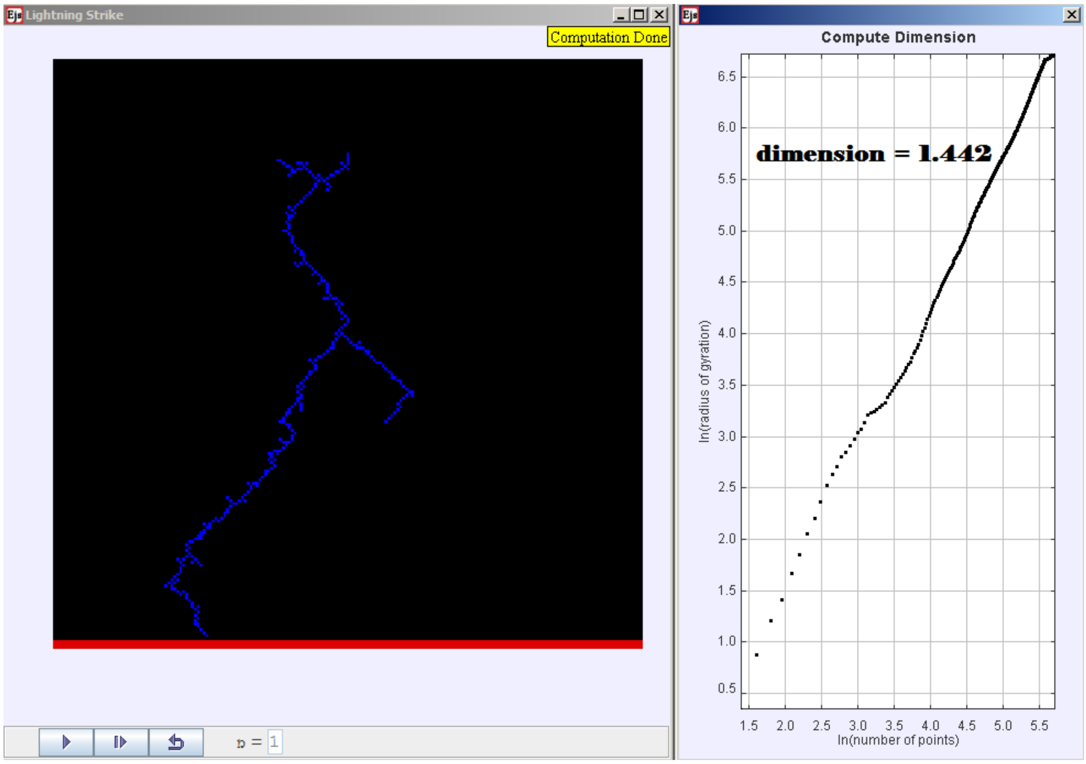

Image 10: DBM lightning strike for η = 1. dimension = 1.442

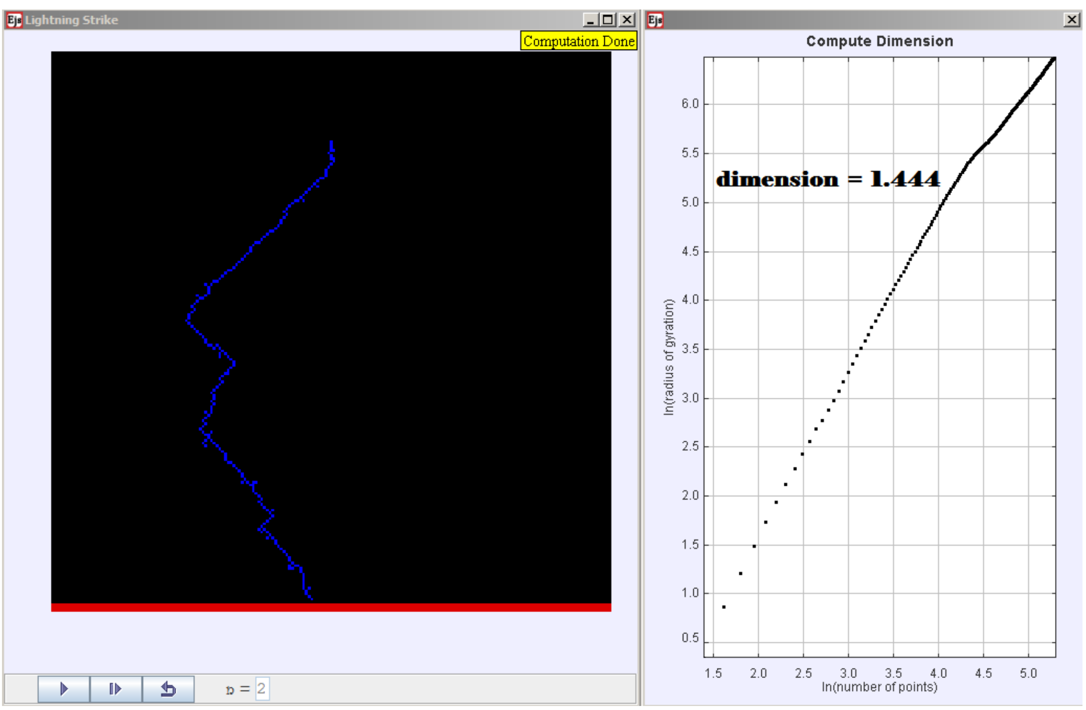

Image 11: DBM lightning strike for η = 2. dimension = 1.444

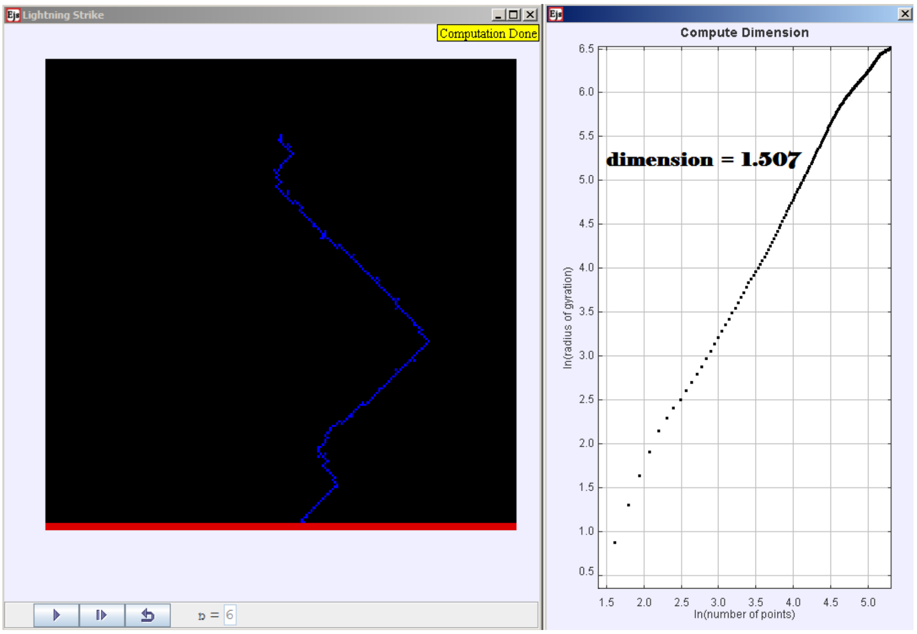

Image 12: DBM lightning strike for η = 6. dimension = 1.507.

Computing the dimension for our DLA simulation, we found that the dimension was generally in the range 1.1-1.2. This is much smaller than expected based on the dimension of the DBM.

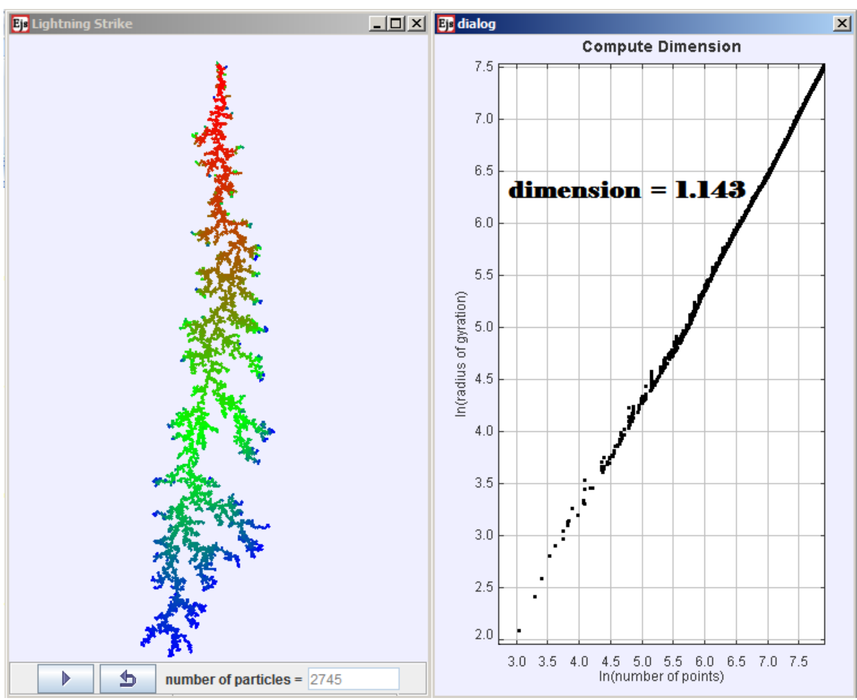

Image 13: DLA lightning strike. dimension = 1.143

<u>Discussion of dimension results:</u>

We found that our modified DLA model for lightning did not produce a fractal dimension that matched the fractal dimension found with the DBM. So, the DLA method we used is probably not a good model for lightning. However, it is also possible that the radius of gyration method is not an appropriate way to compute the fractal dimension for lightning.

To fuel our skepticism, note that the dimension of a line is 1, a plane is 2, and a box has a dimension of 3. It follows that fractals with more branches should have a greater dimension. For our DBM simulation, though, increasing η decreases the amount of branching, but does not decrease the fractal dimension. If anything, we notice that increasing η tends to increase our computed value for the dimension. Perhaps this is a result of the sharp bends commonly taken by the lightning with larger values for η, or perhaps the radius of gyration method is not relevant in this particular instance.

Furthermore, the DLA lightning seems to produce much more branching than the DBM with large values for η. This may simply be caused by discrepancies between the two visualizations, but it is worth investigating the radius of gyration method.

First, to help confirm that our desired computations are working effectively, we added a cursor to the DLA model. This cursor shows the position of the center of mass, which we need to calculate in order to compute dimension. Visually, the center of mass appears correct, located exactly where we expect.

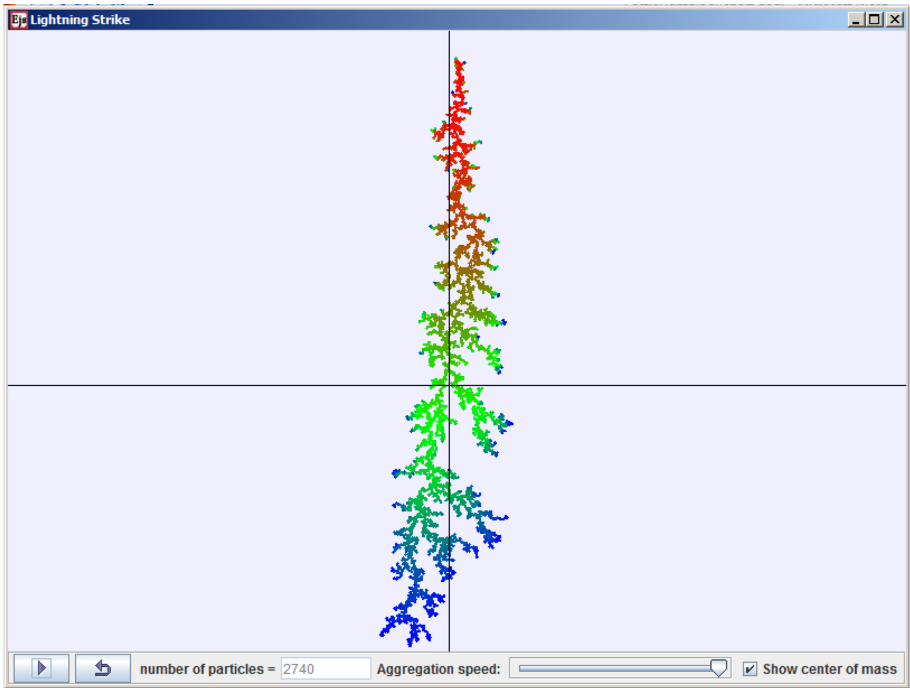

Image 14: DLA model with cursor showing the center of mass

Then, we used our DLA model to generate a standard DLA cluster, such as the one seen in image 7. Now, we really expect the dimension to be larger, but using the radius of gyration method, we computed the dimension of this fractal as still in the range of 1.1-1.2. So, we tried another method for computing fractal dimension; this method is discussed in further detail in reference 8. This method separates the fractal into regions contained within circles centered on the seed particle. As the radius of the circles increases, the total number of particles contained within each circle also increases. On a plot of the natural log of the number of particles contained within the circle versus the natural log of the radius of the circle, the slope is equal to the fractal dimension.

Using this method, we found the fractal dimension for our standard DLA cluster to be approximately 1.5. So, we returned to our DLA lightning model and used this method for computing dimension. Again, we find that the fractal dimension of our DLA lightning is typically in the range 1.1-1.2. This gives more support to the possibility that the dimension of our DLA lightning is actually within this range, but we should still investigate further with other methods for calculating dimension.

Explorations:

We added two options for exploration to our DBM simulation. A check box allows us to run the model with three initial sources of negative charge, representing three separate storm clouds or three concentrated areas of negative potential within a single cloud. When running the simulation with three distinct sources of negative charge, we see that two terminate early and the other continues to the ground. The electrical discharge seems to choose the one quickest path to the ground. This may provide insight to the path of electrical discharge during a lightning strike, or this may be due to the limitations of our model. Further study with multiple initial sources is required.

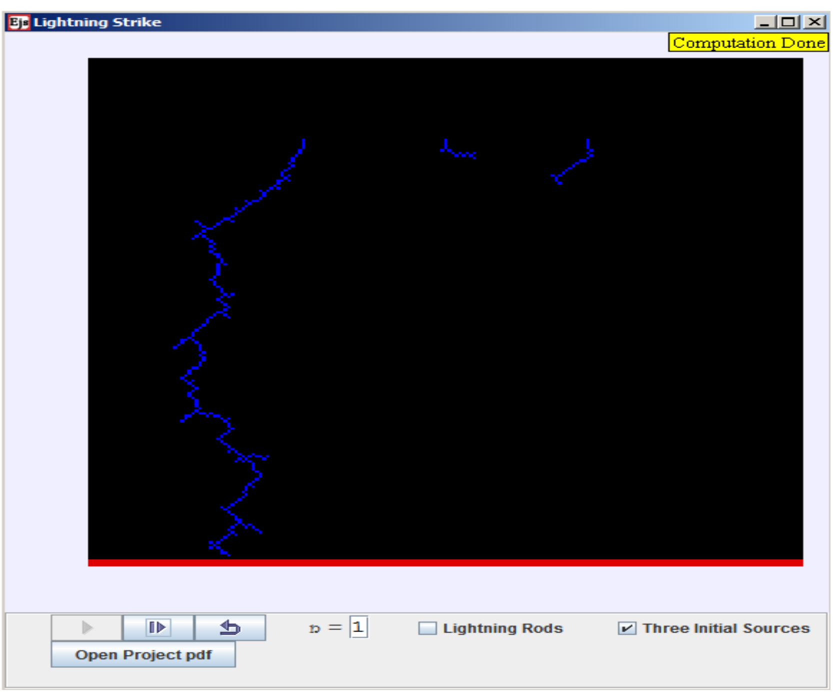

Image 15: DBM lightning (η = 1) from 3 sources of initial negative potential

We can also extend the height of the ground in two different spots. This creates tall conductors of positive potential, emerging from the ground to represent tall buildings or lightning rods. After trying many different configurations, we see that our lightning is much more likely to strike the taller rods. When the rods were about 25% of the height of the lightning source, the lightning bolt rarely preferred to strike the rods. However, when the rods were raised to half the height of the lightning source, the lightning bolt usually struck one of the rods.

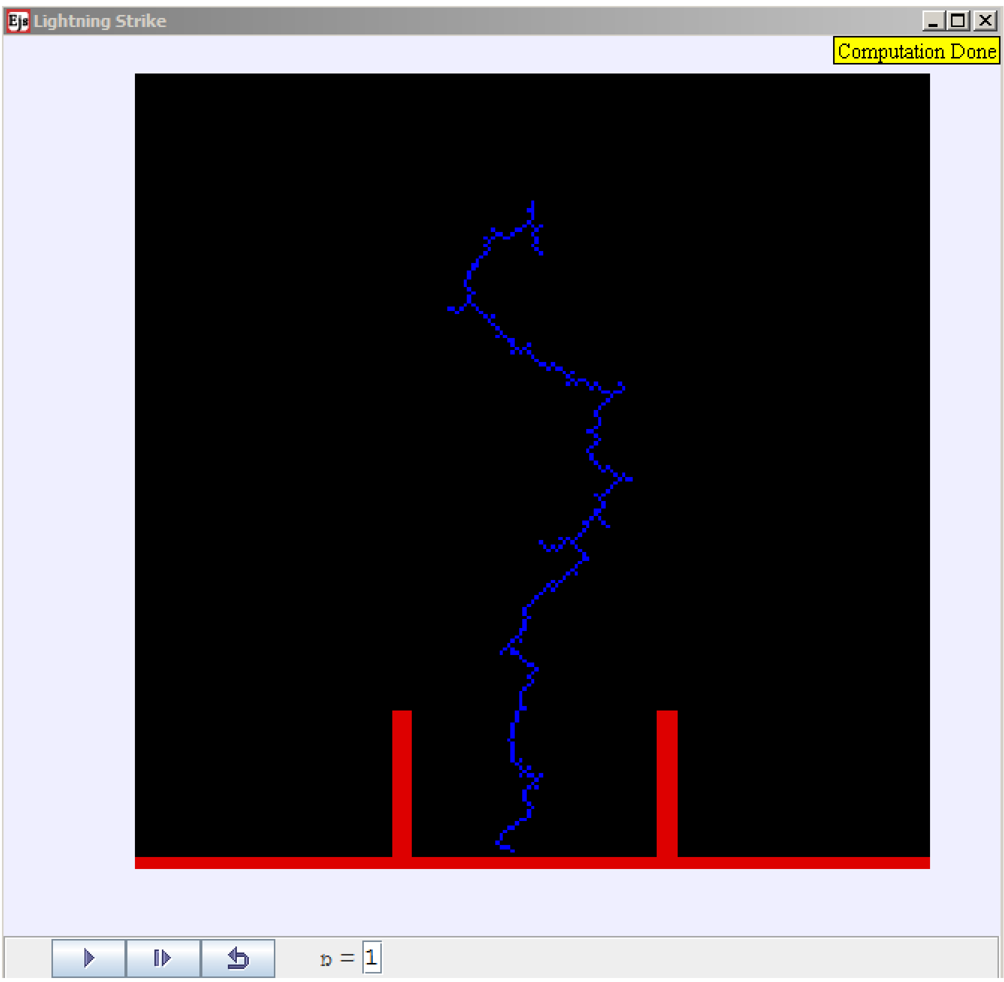

Image 16: DBM lightning (η = 1) with 2 rods, 25% of lightning height

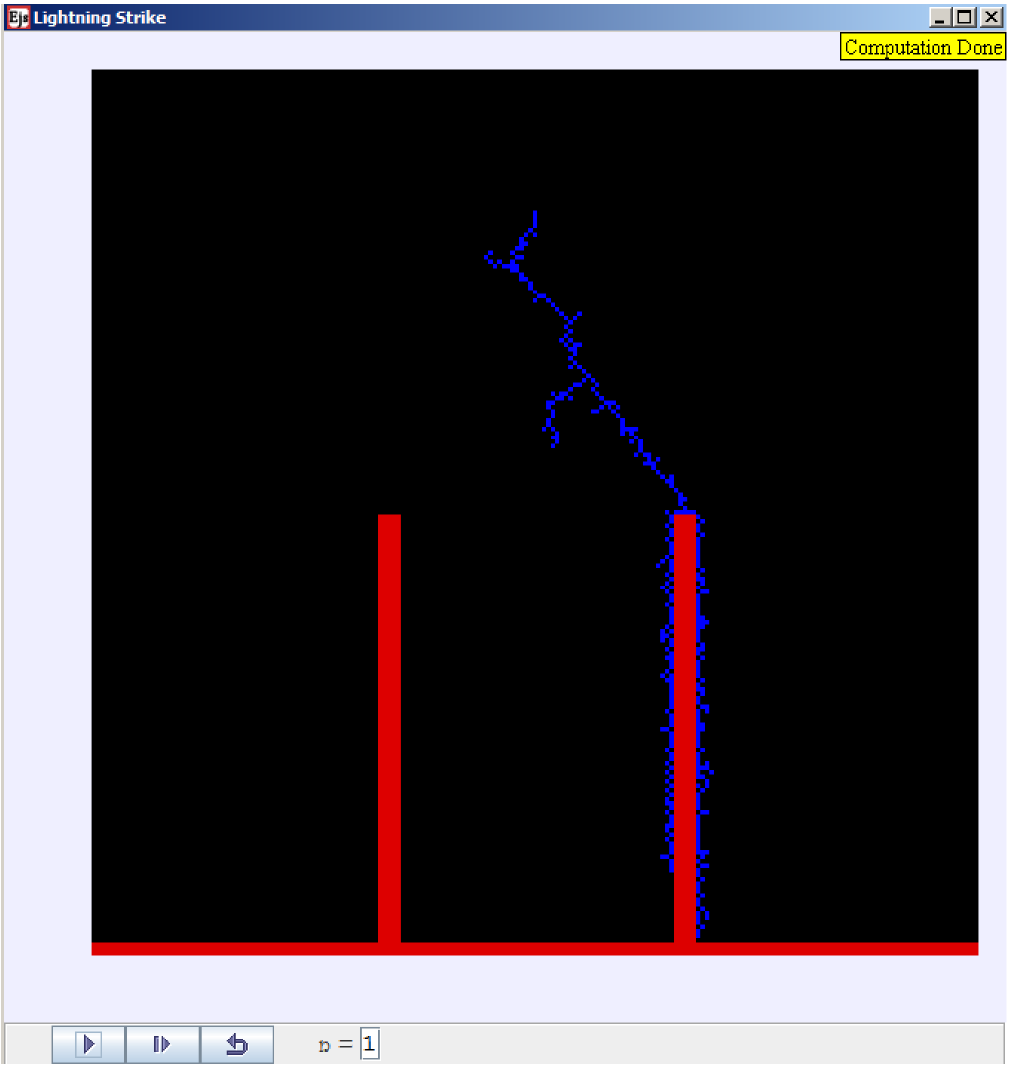

Image 17: DBM lightning (η = 1) with 2 rods, 50% of lightning height

Conclusions:

Our DLA model is likely not an accurate model for lightning, since the calculated dimension is significantly different from the dimension found in the DBM simulation. We are, however, suspicious of our dimension computation, so we should continue to try other numerical methods for calculating the fractal dimension.

It is common knowledge that lightning usually strikes the tallest conductor, but this seems to be the case only if this conductor is significantly higher than the surrounding ground. An opportunity for further research is to conduct a more thorough investigation of the importance of height for lightning rods. We could study a variety of heights for our lightning rods, and for each height, run the simulation a large number of times. The number of times that the lightning hits the rod divided by the total number of times the simulation was run is equal to the probability that our lightning will strike a rod of that height. A plot of probability of a lightning strike versus rod height could provide insight into the optimal height for lightning rods.

Experimenting with different lightning rod shapes could also offer information about their ability to attract lightning strikes. Both of these ideas provide opportunities for further research that could contribute practical information for building effective lightning rods.

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